$A(SL_q(2))$ at roots of unity is a free module over $A(SL(2))$

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Abstract. It is shown that when q is a primitive root of unity of order not equal to 2 mod 4, $A(SL_q(2))$ is a *free* module of finite rank over the coordinate ring of the classical group $SL(2)$. An explicit set of generators is provided.

Quantum groups at roots of unity have particularly rich and interesting structure. In this letter we adopt the 'quantum functions' point of view, dual to the quantum universal enveloping algebras (cf. [7]). It is known that the quantum Hopf algebra $A(SL_q(2))$ at primitive roots of unity $q^l = 1$, l odd, or $q^{2l} = 1$, l even, can be viewed as a module over the algebra $A(SL(2))$, eg. by using the Frobenius mappings [9, 11]. This module is projective and finitely generated [5] and thus it corresponds to a locally free coherent sheaf $\mathcal{F}$ over the affine group scheme $SL(2) =: SL(2, \mathbb{C})$. It has been also established that $SL_q(2)$ at primitive odd roots of unity forms a principal fibre bundle (faithfully flat Hopf-Galois extension) over $SL(2)$ [1, 8]. (See also [3], [4], [6] for the case of cubic root of unity, that was conjectured in [2] to be important in understanding the (quantum) symmetry of the Standard Model of fundamental interactions).

It may have been guessed for a long time that $A(SL_q(2))$ at roots of unity is actually free over $A(SL(2))$ but, as far as we know, no proof has been given. Here we show that it is actually so. Our reasoning underlines the difficulties to tackle the more general problem of other "simple" quantum groups at roots of unity.

Recall that for $\mathbb{C} \ni q \neq 0$, $A(SL_q(2))$ is the free algebra $\mathbb{C}<a,b,c,d>$ modulo the ideal generated by the commutation relations

$$ab = qba \quad ac = qca, \quad bd = qdb, \quad bc = cb, \quad cd = qdc, \quad ad - da = (q - q^{-1})bc$$

and by the q-determinant relation $ad - qbc = 1$. When q is a primitive l-th root of unity, for odd l, the Hopf subalgebra generated by

$$\alpha = a^l, \ \beta = b^l, \ \gamma = c^l, \ \delta = d^l$$

is isomorphic to the Hopf algebra

$$A(SL(2)) = \mathbb{C}[\alpha, \beta, \gamma, \delta] \ / < \alpha\delta - \beta\gamma - 1 >$$

with the restricted coproduct, counit and coinverse. Moreover, $A(SL_q(2))$ is a finitely generated module over $A(SL(2))$. The same happens when l is even and q is a primitive $2l$-th root of unity, but this time $A(SL_q(2))$ has different left and right module structures, due to the appearance of some $\pm$ signs in the commutation relations.

Notice that, using the commutation relations to order monomials, $A(SL_q(2))$ can be generated over $A(SL(2))$ by the l^4 products

$$a^{r_a} b^{r_b} c^{r_c} d^{r_d}, \quad (0 \leq r. \leq l - 1).$$

The determinant relation $ad = 1 + qbc$ reduces the generators above to a smaller set; all the monomials with $r_a r_d > 0$ can be expressed as linear combinations over $\mathbb{C}$ of the following set

$$a^{r_a} b^{r_b} c^{r_c}, \quad b^{r_b} c^{r_c} d^{r_d},$$

of $2l^3$ elements. Indeed

$$P_k(bc) =: a^k d^k = \prod_{j=1}^{k} (1 + q^{2j-1}bc) = \sum_{j=0}^{k} p_{k,j}(q) b^j c^j,$$

where

$$p_{k,j}(q) = q^{j^2} \prod_{r=j+1}^{k} (q^{2r} - 1) \ \left(\prod_{s=1}^{k-j} (q^{2s} - 1) \right)^{-1}.$$

Since $A(SL_q(2))$ is a finitely generated projective module over $A(SL(2))$, it corresponds to a locally free coherent sheaf $\mathcal{F}$ over $Spec(A(SL(2))) = SL(2)$. We now think of $SL(2)$ as a complex manifold $SL(2)_{an}$ instead as an affine scheme.

Proposition 1. $\mathcal{F} \to SL(2)_{an}$ *is locally free of rank* l^3 *and the corresponding vector bundle* F *is trivial.*

Proof. We cover $SL(2)_{an}$ with two open sets U_α, U_β where α or β is invertible. Since $a^{-1} = \alpha^{-1}a^{l-1}$, also a^{-1} exists on U_α and similarly b^{-1} on U_β. Thus on U_α we can set $d = a^{-1}(1 + qbc)$ and use the l^3 elements $a^{r_a}b^{r_b}c^{r_c}$ as generators, and on U_β we set $c = q^{-1}b^{-1}(ad - 1)$ and use $a^{r_a}b^{r_b}d^{r_d}$ as generators.

This proves that $\mathcal{F}$ is locally free of rank l^3 and hence it is a sheaf of sections of a vector bundle F over $SL(2)_{an}$. But $SL(2)_{an}$ retracts to its maximal compact subgroup $SU(2) \simeq S^3$ and since for every Lie group G, $\pi_2(G) = 0$, F is topologically trivial. Now $SL(2)_{an}$ is a Stein domain, the fibre bundles over it have no moduli and are simply classified topologically. It follows that F is also holomorphically trivial. $\qquad\bullet$

The triviality of F (and of $\mathcal{F}$) implies that we should be able to find l^3 global nowhere vanishing independent sections. It is clear that the local trivializations given above are by no means global. To find out global sections of F we work algebraically. The following theorem provides an explicit set of l^3 generators of $A(SL_q(2))$ over $A(SL(2))$.

Proposition 2. *Let* q *be a primitive root of unity of order* l *with* l *odd, or of order* $2l$ *with* l *even. Then* $A(SL_q(2))$ *is a free (left and/or right) module of rank* l^3 *over the coordinate ring of the classical group* $A(SL(2))$. *The* l^3 *generators can be chosen as*

$$a^m b^n c^{s'}, \qquad b^n c^{s''} d^r,$$

with the integers m, n, r, s', s'' *in the range* $1 \le m \le l-1$; $0 \le n, r \le l-1$; $m \le s' \le l-1$; $0 \le s'' \le l-r-1$.

Proof. For concreteness, we choose the left module structure, the right module case is analogous. Using the above expression for $P_k(bc)$ and noticing that $p_{k,0}(q) = 1$, one has

$$b^r c^s d^{l-k} = q^{(l-k)(r+s)} \, \delta \, a^k b^r c^s - \sum_{j=1}^{k} p_{k,j}(q) \, b^{r+j}c^{s+j}d^{l-k},$$

$$a^{l-k}b^r c^s = q^{-k(r+s)} \, \alpha \, b^r c^s d^k - \sum_{j=1}^{k} p_{k,j}(q) \, a^{l-k}b^{r+j}c^{s+j}.$$

The idea is to use recursively these relations to eliminate the monomials in the l.h.s.'s in terms of those in the r.h.s.'s. Of course there is a cyclicity problem, because the recursive substitutions will sooner or later bring in the r.h.s the monomial in the l.h.s. multiplied by a non constant element in $A(SL(2))$. We then have to put bounds on the ranges of the exponents. Let us look at the first equation. As d occurs with the same exponent in both sides, there are no restrictions on k. Fix $s = s_1$ (possibly depending on k), let $S_1(k) = \{(s_1 + j) \bmod l, \ 0 \le j \le k\}$, $S_1^c(k) = \mathbb{Z}_l - S_1(k)$. We can use the

monomials $b^r c^s d^{l-k-1}$ with $s \in S_1(k)$, $k, r \in \mathbb{Z}_l$ and $a^k b^r c^s$ with $k, r, s \in \mathbb{Z}_l$ to generate the monomials in the l.h.s. with $s \in S_1^c(k)$. The same can be done for the second equation, by choosing s_2. We get a system of generators of the form $b^r c^s d^{l-k-1}$, with $r, k \in \mathbb{Z}_l$, $s \in S_1(k)$, and $a^k b^r c^s$, with $k \in \mathbb{Z}_l - \{0\}$, $r \in \mathbb{Z}_l$ and $s \in S_2(l - k - 1)$. These are l^3 elements and we have to choose carefully s_1, s_2 to be sure that there are no relations among them. This is the case for $s_1 = 0$ and $s_2 = k$. By a suitable relabelling of the indices, we complete the proof of the theorem. $\bullet$

Remark 1. The results above hold for q being a primitive root of unity of order not equal to 2 mod 4. In the remaining cases, $q^{2l} = 1$, l odd, the $2l$-th powers of a, b, c, d commute but neither the determinant relation nor the coproduct close on the algebra generated by them. To remedy this mismatch one can enlarge the algebra by taking α, β, γ, δ (the l-th powers of a, b, c, d) as generators, obtaining however a non-commutative algebra.

Remark 2. In attempting to generalize the above results to $SL_q(n)$ we immediately face the lack of the topological argument used in the proof of Proposition 1, which in fact has been the starting point of this letter. So in principle one has to use algebra alone. Local triviality easily follows by considering the open covering of $SL(n)$ given by the subsets U_k where the algebraic complements $\hat{T}_{k1}$ of the elements of the first column are invertible (see [10] for notations). On U_k we have

$$T_{k1} = (-q)^{1-k}[1 - \sum_{j \neq k}(-q)^{j-1}T_{j1}\hat{T}_{j1}]\hat{T}_{k1}^{-1}$$

and a basis of the localized module is given by the l^{n^2-1} monomials $\prod_{(r,s) \neq (k,1)} T_{rs}^{n_{rs}}$, with $n_{rs} \in \mathbb{Z}_l$. This simplifies in the special case of $SL_q(n)$ the proof in [5] valid for a general quantum group. The determinant relation provides equations similar to those of Proposition 2, but the elimination problem is much subtler.

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